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# On the equivalence of 'non-equivalent' algebraic realizations

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**Abstract**

Symmetries have been used with great success to determine solutions of differential equations with which they are associated. In addition to reducing the order of the equation, one can also use the Lie algebra of the symmetries to transform the equation into 'canonical form'. The canon, in this case, is determined by the 'standard' realization of the Lie algebra. Following a comment in Olver P J (1995 *Equivalence, Invariants and Symmetry* (Cambridge: Cambridge University Press)), we conjecture that while Lie algebras may have non-equivalent realizations in the usual (point transformation) sense, *all* realizations of the same Lie algebra are equivalent when considered on the appropriate ordered jet space. We show how this result can have useful implications for ordinary differential equations, including linearization for equations thought to be inherently nonlinear.

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## 1. Introduction

Sophus Lie made the remarkable observation that most differential equations solvable via different (usually *ad hoc*) techniques had in common the property of invariance under a (usually but not necessarily) point transformation. He exploited this property to develop a method to treat (almost) all differential equations in a similar manner to Galois' approach for algebraic equations. In spite of not fully realizing his aim, his technique has proven to be extremely useful in finding (invariably physically relevant) solutions to differential equations.

In order to utilize this beneficial property of differential equations, one looks for symmetries which generate the necessary invariant transformations. Once this is achieved, the symmetries are usually used to reduce the order of the equation (hopefully to a first order equation which can be easily solved). The interested reader is referred to [1] and references therein for full details.

**Table 1.** Second-order equations admitting 2D Lie algebras.

| Type | $[G_1, G_2]$ | Symmetries | | Invariant equation |
|---|---|---|---|---|
| I | 0 | $G_1 = \partial_x$ | $G_2 = \partial_u$ | $u_{xx} = F(u_x)$ |
| II | 0 | $G_1 = \partial_u$ | $G_2 = x\partial_u$ | $u_{xx} = F(x)$ |
| III | $G_1$ | $G_1 = \partial_u$ | $G_2 = x\partial_x + u\partial_u$ | $xu_{xx} = F(u_x)$ |
| IV | $G_1$ | $G_1 = \partial_u$ | $G_2 = u\partial_u$ | $u_{xx} = u_x F(x)$ |

Another important use of symmetries of differential equations is the transformation of equations into 'canonical' form. As an example, let us consider a second-order differential equation, $E$ say, which is invariant under a 2D Lie algebra of symmetries. If the Lie algebra is Abelian, i.e. the Lie bracket of the symmetries commute, then there are two 'inequivalent' canonical realizations of the Lie algebra—types I and II of table 1. This implies that this equation, $E$, can only be transformed into one of the corresponding equations under a point transformation. Indeed, the two equations invariant under the 2D Abelian Lie algebra cannot be transformed into one another via a point transformation as their respective symmetries cannot be transformed into each other under a point transformation. As a result, one usually states that the two sets of symmetries constitute 'inequivalent' realizations of the same Lie algebra.

In this paper, we show that the usual concept of 'inequivalent' realizations of Lie algebras arises as a result of purely point transformation considerations. If the class of transformation is allowed to lie in an appropriately ordered ($n \geqslant 1$) jet space, then *all* realizations of Lie algebras are equivalent. (While a formal proof is not provided, we indicate why we believe that this conclusion can be reached by extrapolation from simple examples.) In order to demonstrate this, we need to prolong the usual vector fields (symmetries of the differential equations) to arbitrary order. In general we work on an open subset $M \subset X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$ of the space of $p$ independent and $q$ dependent variables. $G$ will be a connected $r$-dimensional local transformation group acting on this subset—this action induces an action on the $n$th order jet bundle $\mathrm{J}^n = \mathrm{J}^n M$ denoted by $G^{(n)}$, the $n$th prolongation of $G$.

The action of $G$ is generated by a vector field given by

$$\mathbf{X} = \sum_{i=1}^{p} \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^\alpha}, \tag{1}$$

while the induced action on $\mathrm{J}^n$ is generated by the $n$th order prolongation of (1) given by

$$\mathrm{pr}^{(n)} X = X + \sum_{\alpha=1}^{q} \sum_{J} \phi_\alpha^J(\mathbf{x}, \mathbf{u}^{(n)}) \frac{\partial}{\partial u_J^\alpha}. \tag{2}$$

Here the second summation is over all multi-indices $J = (j_1, \ldots, j_\ell)$ with $1 \leqslant j_\ell \leqslant p$, $1 \leqslant \ell \leqslant n$ and the coefficient functions in (2) are given by [1]

$$\phi_\alpha^J(\mathbf{x}, \mathbf{u}^{(n)}) = D_J \left( \phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha \right) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha. \tag{3}$$

Here we confine ourselves to the Euclidean case (one independent and one dependent variable).

The plan of our paper is as follows. Firstly, we answer a question in [2, p 121] by showing that all actions (real and complex) of SL(2) are equivalent. Thereafter we show that the same result is true for the 2D Lie algebras in table 1. Finally we indicate why these results should hold true in general and consider the implications of these results for differential equations.

**Table 2.** Locally inequivalent actions of $SL(2)$.

| Type | Generators/symmetries | | |
|---|---|---|---|
| I | $G_1 = \partial_x$ | $G_2 = x\partial_x$ | $G_3 = x^2\partial_x$ |
| II | $G_1 = \partial_x$ | $G_2 = x\partial_x - u\partial_u$ | $G_3 = x^2\partial_x - 2xu\partial_u$ |
| III | $G_1 = \partial_x$ | $G_2 = x\partial_x - u\partial_u$ | $G_3 = x^2\partial_x - (2xu+1)\partial_u$ |
| IV | $G_1 = \partial_x$ | $G_2 = x\partial_x + u\partial_u$ | $G_3 = (x^2 - u^2)\partial_x + 2xu\partial_u$ |

## 2. Equivalence of $SL(2)$ actions

Olver [2] recently posed the question of the equivalence of different (complex) actions of a given transformation group via prolongation. This was motivated by his observance that the three locally inequivalent actions of $SL(2)$ on a two-dimensional complex manifold could be related by this simple process.

If we take the first of the three (complex) actions in table 2 and prolong each of the symmetries to first order, we obtain

$$G_{11}^{[1]} = \partial_x \tag{4}$$

$$G_{12}^{[1]} = x\partial_x - p\partial_p \tag{5}$$

$$G_{13}^{[1]} = x^2\partial_x - 2xp\partial_p, \tag{6}$$

where $p = u_x$ and $G_{ij}^{[n]}$ refers to the $n$th prolongation of the $j$th symmetry in the type I group action of table 2. We observe that (4)–(6) have the same form as the symmetries of the type II group action, i.e. the transformation $(x, u) \mapsto (x, u_x)$ maps the type I action to the type II action. The presence of $u_x$ in the transformation ensures that we are not dealing with a point transformation but rather a generalized transformation.

The second prolongation of the type I symmetries is given by

$$G_{11}^{[2]} = \partial_x \tag{7}$$

$$G_{12}^{[2]} = x\partial_x - p\partial_p - 2q\partial_q \tag{8}$$

$$G_{13}^{[2]} = x^2\partial_x - 2xp\partial_p - (4xq + 2p)\partial_q, \tag{9}$$

where $q = u_{xx}$. The transformation $(x, u) \mapsto (x, u_{xx}/(2u_x))$ maps the symmetries (7)–(9) to those of the type III action. These results all appear in [2, 3].

The final action in table 2 is a real action and Olver [2] wondered if this too was equivalent to the other complex actions, i.e., could it be obtained from the same source. Indeed this is the case. It can be shown that the transformation

$$(x, u) \mapsto \left( x + \frac{2u_x u_{xx}}{u_x^4 + u_{xx}^2}, \frac{2u_x^3}{u_x^4 + u_{xx}^2} \right) \tag{10}$$

maps the symmetries (7)–(9) to those of the type IV action. (Note that the general transformation has the form

$$(x, u) \mapsto \left( x + \frac{2\alpha^2(w + \beta)}{u_x(4 + \alpha^2(w + \beta)^2)}, \frac{4\alpha}{u_x(4 + \alpha^2(w + \beta)^2)} \right), \tag{11}$$

where $w = u_{xx}\big/u_x^2$ and $\alpha$ and $\beta$ are arbitrary constants set to two and zero respectively to obtain (10).) These transformations are summarized in table 3 which also includes some higher order derivatives.

**Table 3.** Transformations between SL(2) actions.

| Type | $x$ | $u$ | $u_x$ | $u_{xx}$ |
|---|---|---|---|---|
| I $\to$ II | $x$ | $u_x$ | $u_{xx}$ | $u_{xxx}$ |
| I $\to$ III | $x$ | $\frac{u_{xx}}{2u_x}$ | $-\frac{u_{xx}^2-u_xu_{3x}}{2u_x^2}$ | $\frac{2u_{xx}^3-3u_xu_{xx}u_{3x}+u_x^2u_{4x}}{2u_x^3}$ |
| I $\to$ IV | $x+\frac{2u_xu_{xx}}{u_x^4+u_{xx}^2}$ | $\frac{2u_x^3}{u_x^4+u_{xx}^2}$ | $-\frac{2u_x^2u_{xx}}{u_x^4-u_{xx}^2}$ | $-\frac{2u_x\left(u_x^4+u_{xx}^2\right)^3\left(u_xu_{3x}-2u_{xx}^2\right)}{\left(u_x^4-u_{xx}^2\right)^3\left(u_x^4+2u_xu_{3x}-3u_{xx}^2\right)}$ |

## 3. Equivalence of the 2D actions

We now turn our attention to the two Lie algebras represented in table 1. In the case of the Abelian Lie algebra we use the type II action as our starting point (since the $n$th prolongations of the symmetries for the type I action are identical to the nonprolonged symmetries). Using the first prolongation we find that the transformation which maps the symmetries of the type II action with those of the type I action is $(x,u)\mapsto(u-xu_x+f(x),u_x+g(x))$, where $f$ and $g$ are arbitrary functions usually set to zero. (Note that, as the symmetries commute, the transformations for $x$ and $u$ can be interchanged.)

In the case of the non-Abelian Lie algebra we take the first prolongation of the symmetries of the type IV action and obtain the transformation $(x,u)\mapsto(f(x)u_x,u+g(x)u_x)$ which maps them to the symmetries of the type III action. (Again $f$ and $g$ are arbitrary functions, in this case, usually taken to be one and zero respectively.)

## 4. General equivalence

In order to understand the equivalence of different realizations of Lie algebras, we need to scrutinize the transformation equations more closely. Let us revisit the 2D non-Abelian Lie algebra. In order to transform the type IV action (in variables $x$ and $u$) to the type III action (in variables $X$ and $U$), we usually search for a transformation of the form

$$X=F(x,u)\qquad U=G(x,u)\tag{12}$$

and so ensure that we are looking for a point transformation. The relevant transformation equations are

$$F_u\partial_X+G_u\partial_U=\partial_U\tag{13}$$

$$u(F_u\partial_X+G_u\partial_U)=F\partial_X+G\partial_U.\tag{14}$$

The first equation yields

$$F=f(x)\qquad G=u+g(x).\tag{15}$$

Substituting this into the second equation leaves us with

$$u\partial_U=f(x)\partial_X+(u+g(x))\partial_U.\tag{16}$$

While we can set $g(x)$ to zero without any problems, the requirement that $f(x)$ must also be zero makes the transformation (12) meaningless. It is clear that we require an additional term in (16) to 'balance' the $f(x)$ term. One way of achieving this is to enlarge the class of transformation to include higher order derivatives. Merely allowing $F$ and $G$ to include a dependence on $u_x$ and then taking the first prolongation of the symmetries of the type IV action is sufficient: (12) now becomes

$$X = F(x, u, u_x) \qquad U = G(x, u, u_x) \tag{17}$$

and we use the first prolongations

$$G_{11}^{[1]} = \partial_u \tag{18}$$

$$G_{21}^{[1]} = u\partial_u + u_x\partial_{u_x}. \tag{19}$$

The new transformation equations are

$$F_u\partial_X + G_u\partial_U = \partial_U \tag{20}$$

$$u(F_u\partial_X + G_u\partial_U) + u_x\left(F_{u_x}\partial_X + G_{u_x}\partial_U\right) = F\partial_X + G\partial_U. \tag{21}$$

This time, the first equation yields

$$F = F(x, u_x) \qquad G = u + g(x, u_x). \tag{22}$$

Now (21) becomes

$$u\partial_U + u_x(F_{u_x}\partial_X + g_{u_x}\partial_U) = F\partial_X + (u+g)\partial_U. \tag{23}$$

Here we can 'balance' all terms and obtain the transformation

$$(x, u) \mapsto (f(x)u_x, u + g(x)u_x) \tag{24}$$

between the type IV and type III 2D Lie algebras.

This is not always the case as is evidenced by the presence of second derivatives in two of the transformations in the case of $SL(2)$. However, we can always ensure that the functions in (12) are made general enough to include suitable $n$th order derivatives. By taking appropriate $n$th prolongations of the symmetries, we will be able to map the symmetries of the different realizations of the same Lie algebra to each other.

## 5. Discussion

We have indicated how 'different realizations' (i.e. not equivalent under a point transformation) of the same Lie algebra can be made to be equivalent by extending our class of transformations to the appropriate ordered jet space. That this non-equivalence at the point transformation level is removed by the above extension is not that surprising. An examination of the methods used to determine the different realizations of a particular Lie algebra reveals that the calculations are usually restricted to functions of independent and dependent variables (see, e.g., [4]). As a result, it is not surprising that these 'different realizations' can be made equivalent by appropriate extensions. Thus we have the following conjecture: all actions of a particular group are equivalent when these actions are prolonged to the appropriate $n$th ordered jet space $J^n$.

In addition to being an interesting theoretical result, the above conjecture has useful direct applications. This has already been seen in the remarkable results of [3]. Here we point to useful consequences in the case of the 2D Lie algebras.

Let us revisit to the 2D Lie algebras. The equivalence transformations quoted above were obtained purely from the algebras themselves. As a result, we did not place a restriction on their forms. However, if we want to ensure that the equations are also transformed appropriately (i.e. that the order is unchanged), it turns out that the general transformations quoted in

**Table 4.** Transformations between 2D actions.

| Type | $\tilde{x}$ | $\tilde{u}$ | $\tilde{u}_{\tilde{x}}$ | $\tilde{u}_{\tilde{x}\tilde{x}}$ | Restriction |
|---|---|---|---|---|---|
| II $\rightarrow$ I | $u - xu_x + f(x)$ | $u_x + g(x)$ | $-\dfrac{1}{x}$ | $\dfrac{1}{x^2(f_x - xu_{xx})}$ | $f_x = -xg_x$ |
| IV $\rightarrow$ III | $f(x)u_x$ | $u + g(x)u_x$ | $\dfrac{g}{f}$ | $-\dfrac{1}{f(f_x u_x + f u_{xx})}$ | $\dfrac{1+g_x}{g} = \dfrac{f_x}{f}$ |

section 3 are indeed too general. To ensure that the transformations are appropriate for the second-order equations in table 1, i.e. that the transformed equations are again of second order, we must impose some restrictions. Drawing from results in contact transformations, we insist that the first derivatives of the transformation are free of second derivatives. Thus for the mapping from the type II case to that of the type I case, namely

$$(\tilde{x}, \tilde{u}) \mapsto (u - xu_x + f(x), u_x + g(x)), \tag{25}$$

we must investigate $\mathrm{d}\tilde{u}/\mathrm{d}\tilde{x}$. Here we find that

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tilde{x}} = \frac{u_{xx} + g_x}{-xu_{xx} + f_x} \tag{26}$$

and so it can, at most, be a function of $x$ only. This forces the relationship

$$f_x = -xg_x. \tag{27}$$

Thus, the 'usual' practice of setting $f(x) = g(x) = 0$ will work here. We now have the important result that the linear equation

$$u_{xx} = F(x) \tag{28}$$

and the *nonlinear* equation

$$\tilde{u}_{\tilde{x}\tilde{x}} = \tilde{F}(\tilde{u}_{\tilde{x}}) \tag{29}$$

are related via (25) provided (27) holds.

Let us now consider the mapping from the type IV case to the type III case, namely

$$(\tilde{x}, \tilde{u}) \mapsto (f(x)u_x, u + g(x)u_x). \tag{30}$$

The first derivative is

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tilde{x}} = \frac{u_x + g_x u_x + g u_{xx}}{f_x u_x + f u_{xx}} \tag{31}$$

which, it turns out, can also only be a function of $x$. As a result, the relationship

$$\frac{1 + g_x}{g} = \frac{f_x}{f} \tag{32}$$

must be satisfied. Here, the 'usual' practice of setting $f(x) = 1$ and $g(x) = 0$ will *not* work. We now have that the linear equation

$$u_{xx} = u_x F(x) \tag{33}$$

is related to the *nonlinear* equation

$$\tilde{x}\tilde{u}_{\tilde{x}\tilde{x}} = \tilde{F}(\tilde{u}_{\tilde{x}}) \tag{34}$$

via (30) provided (32) holds. We summarize these transformations in table 4.

Thus, the equivalence of 'inequivalent' realizations of the same Lie algebra can be usefully utilized to transform differential equations in each other. Here we have shown how two families of nonlinear equations could be linearized. Clearly similar results can be obtained for other Lie algebras. This work is ongoing.

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